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Fixed-Point Theorems for Multivalued Maps with Closed Values on Complete Gauge Spaces

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Abstract—In this note, we present fixed-point results for contractive maps in the sense of Bose and Mukherjee defined on complete gauge spaces.

Keywords—Fixed-point theorems, Multivalued maps, Complete gauge space, Pseudometric and completeness.

1. INTRODUCTION

In this paper, we present new fixed-point results for multivalued maps with closed values defined on a complete gauge space. The ideas needed are elementary and only rely on the notions of pseudometric and completeness. Our results in particular extend those of Agarwal and O'Regan [1], Bose and Mukherjee [2], and Frigon [3].

For the remainder of this section, we present some notations which will be used in Section 2. Throughout this paper, $X = (X, \{d_\alpha\}_{\alpha\in\Lambda})$ will denote a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$ (see [4, pp. 198, 308]). For $A \subseteq X$ and $x \in X$ fixed, by $\mathrm{dist}_\alpha(x, A)$ we mean $d_\alpha(x, A)$. For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $x \in X$, we define the pseudo-ball centered at x of radius r by

$$B(x,r) = \{y \in X : d_\alpha(x,y) \le r_\alpha \text{ for all } \alpha \in \Lambda\}.$$

We denote by D_α the generalized Hausdorff pseudometric induced by d_α; that is, for $Z, Y \subseteq X$,

$$D_\alpha(Z,Y) = \inf\{\epsilon > 0 : \forall\, x \in Z,\ \forall\, y \in Y,\ \exists\, x^\star \in Z,\ \exists\, y^\star \in Y \text{ such that } d_\alpha(x,y^\star) < \epsilon,\ d_\alpha(x^\star,y) < \epsilon\},$$

with the convention that $\inf(\emptyset) = \infty$.

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2. FIXED-POINT RESULTS

This section presents new fixed-point results for multivalued maps with closed values defined on a complete gauge space X. Our main theorems generalize and extend results in [1–3].

THEOREM 2.1. *Let X be a complete gauge space, $r \in (0,\infty)^\Lambda$, $x_0 \in X$, and $F : B(x_0, r) \to C(X)$; here $C(X)$ denotes the family of nonempty closed subsets of X. Suppose there exist constants $a_1 = \{a_{1,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_2 = \{a_{2,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_3 = \{a_{3,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_4 = \{a_{4,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, and $a_5 = \{a_{5,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ such that for every $\alpha \in \Lambda$ and every $x, y \in B(x_0, r)$, we have*

$$\begin{aligned} D_\alpha(F(x), F(y)) \le{}& a_{1,\alpha} \operatorname{dist}_\alpha(x, F(x)) + a_{2,\alpha} \operatorname{dist}_\alpha(y, F(y)) + a_{3,\alpha} \operatorname{dist}_\alpha(y, F(x)) \\ &+ a_{4,\alpha} \operatorname{dist}_\alpha(x, F(y)) + a_{5,\alpha}\, d_\alpha(x,y), \end{aligned}$$

where for every $\alpha \in \Lambda$, $a_{1,\alpha}+a_{2,\alpha}+a_{3,\alpha}+a_{4,\alpha}+a_{5,\alpha} < 1$, $a_{1,\alpha}+a_{4,\alpha}+a_{5,\alpha} > 0$, $a_{2,\alpha}+a_{3,\alpha}+a_{5,\alpha} > 0$, and with either $a_{1,\alpha} = a_{2,\alpha}$ or $a_{3,\alpha} = a_{4,\alpha}$. In addition, assume the following two conditions hold:

$$\begin{aligned} &\text{for every } x \in B(x_0,r) \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \text{ there exists} \\ &y \in F(x) \text{ with } d_\alpha(x,y) \le \operatorname{dist}_\alpha(x, F(x)) + \epsilon_\alpha \text{ for every } \alpha \in \Lambda \end{aligned} \tag{2.1}$$

and

$$\text{for } \alpha \in \Lambda, \qquad \operatorname{dist}_\alpha(x_0, F(x_0)) < \left(1 - \frac{A_{1,\alpha}(1 + A_{2,\alpha})}{1 + A_{1,\alpha}}\right) r_\alpha, \tag{2.2}$$

where

$$A_{1,\alpha} = \frac{a_{1,\alpha} + a_{4,\alpha} + a_{5,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}} \qquad \text{and} \qquad A_{2,\alpha} = \frac{a_{2,\alpha} + a_{3,\alpha} + a_{5,\alpha}}{1 - a_{1,\alpha} - a_{3,\alpha}}.$$

Then F has a fixed point (i.e., there exists $x \in B(x_0, r)$ with $x \in F(x)$).

REMARK 2.1. Fix $\alpha \in \Lambda$. Note if $a_{3,\alpha} = a_{4,\alpha}$, then $0 < A_{1,\alpha} < 1$ and $0 < A_{2,\alpha} < 1$, whereas, if $a_{1,\alpha} = a_{2,\alpha}$, we have $0 < A_{1,\alpha} A_{2,\alpha} < 1$. We may if we wish (because of symmetry) take $a_{3,\alpha} = a_{4,\alpha}$ and $a_{1,\alpha} = a_{2,\alpha}$ (in this case $A_{1,\alpha} = A_{2,\alpha}$).

PROOF. From (2.1) and (2.2), we may choose a $x_1 \in F(x_0)$ with

$$d_\alpha(x_1, x_0) < (1 - k_\alpha)\, r_\alpha, \qquad \text{for every } \alpha \in \Lambda; \tag{2.3}$$

here

$$k_\alpha = 1 - \frac{A_{1,\alpha}(1 + A_{2,\alpha})}{1 + A_{1,\alpha}}. \tag{2.4}$$

Notice $x_1 \in B(x_0, r)$.

Next for $\alpha \in \Lambda$, choose $\epsilon_\alpha > 0$ such that

$$A_{1,\alpha}\, d_\alpha(x_1, x_0) + \frac{\epsilon_\alpha}{1 - a_{2,\alpha} - a_{4,\alpha}} < A_{1,\alpha}\,(1 - k_\alpha)\, r_\alpha. \tag{2.5}$$

Then choose $x_2 \in F(x_1)$ so that for every $\alpha \in \Lambda$, we have

$$\begin{aligned} d_\alpha(x_1, x_2) &\le \operatorname{dist}_\alpha(x_1, F(x_1)) + \epsilon_\alpha \\ &\le D_\alpha(F(x_0), F(x_1)) + \epsilon_\alpha \\ &\le a_{1,\alpha} \operatorname{dist}_\alpha(x_0, F(x_0)) + a_{2,\alpha} \operatorname{dist}_\alpha(x_1, F(x_1)) + a_{3,\alpha} \operatorname{dist}_\alpha(x_1, F(x_0)) \\ &\quad + a_{4,\alpha} \operatorname{dist}_\alpha(x_0, F(x_1)) + a_{5,\alpha}\, d_\alpha(x_0, x_1) + \epsilon_\alpha \\ &\le a_{1,\alpha}\, d_\alpha(x_0, x_1) + a_{2,\alpha}\, d_\alpha(x_1, x_2) + a_{4,\alpha}\, d_\alpha(x_0, x_2) + a_{5,\alpha}\, d_\alpha(x_0, x_1) + \epsilon_\alpha, \end{aligned}$$

and so

$$d_\alpha(x_1, x_2) \le A_{1,\alpha}\, d_\alpha(x_0, x_1) + \frac{\epsilon_\alpha}{1 - a_{2,\alpha} - a_{4,\alpha}}.$$

Then because of (2.5) we have chosen $x_2 \in F(x_1)$ so that

$$d_\alpha(x_1, x_2) < A_{1,\alpha}\,(1-k_\alpha)\,r_\alpha, \qquad \text{for every } \alpha \in \Lambda. \tag{2.6}$$

Notice

$$x_2 \in B(x_0, r),$$

since (we give an argument here which can be used in the general step) for $\alpha \in \Lambda$, we have

$$\begin{aligned} d_\alpha(x_0, x_2) &\le (1-k_\alpha)\,r_\alpha + A_{1,\alpha}\,(1-k_\alpha)\,r_\alpha \\ &\le (1-k_\alpha)\,r_\alpha\,\big\{1 + A_{1,\alpha}\,A_{2,\alpha} + (A_{1,\alpha}\,A_{2,\alpha})^2 + \cdots \\ &\quad + A_{1,\alpha}\,\left[1 + A_{1,\alpha}\,A_{2,\alpha} + (A_{1,\alpha}\,A_{2,\alpha})^2 + \cdots\right]\big\} \\ &= (1-k_\alpha)\,r_\alpha \left[\frac{1 + A_{1,\alpha}}{1 - A_{1,\alpha}\,A_{2,\alpha}}\right] = r_\alpha. \end{aligned}$$

Next for $\alpha \in \Lambda$, choose $\delta_\alpha > 0$ such that

$$A_{2,\alpha}\,d_\alpha(x_1, x_2) + \frac{\delta_\alpha}{1 - a_{1,\alpha} - a_{3,\alpha}} < A_{2,\alpha}\,A_{1,\alpha}\,(1-k_\alpha)\,r_\alpha. \tag{2.7}$$

Then choose $x_3 \in F(x_2)$ so that for every $\alpha \in \Lambda$, we have

$$d_\alpha(x_2, x_3) \le \operatorname{dist}_\alpha(x_2, F(x_2)) + \delta_\alpha.$$

A similar reasoning as above yields for every $\alpha \in \Lambda$ that

$$d_\alpha(x_2, x_3) \le A_{2,\alpha}\,d_\alpha(x_1, x_2) + \frac{\delta_\alpha}{1 - a_{1,\alpha} - a_{3,\alpha}},$$

and so

$$d_\alpha(x_3, x_2) < A_{2,\alpha}\,A_{1,\alpha}\,(1-k_\alpha)\,r_\alpha, \qquad \text{for every } \alpha \in \Lambda. \tag{2.8}$$

Notice

$$x_3 \in B(x_0, r),$$

since for $\alpha \in \Lambda$ we have (see the reasoning above)

$$\begin{aligned} d_\alpha(x_0, x_3) &\le (1-k_\alpha)\,r_\alpha\,[1 + A_{1,\alpha} + A_{1,\alpha}\,A_{2,\alpha}] \\ &\le (1-k_\alpha)\,r_\alpha \left[\frac{1 + A_{1,\alpha}}{1 - A_{1,\alpha}\,A_{2,\alpha}}\right] = r_\alpha. \end{aligned}$$

Proceed inductively to obtain $x_n \in F(x_{n-1})$, $n = 4, 5, \ldots$ with $x_n \in B(x_0, r)$ and

$$d_\alpha(x_{2j+1}, x_{2j+2}) \le (A_{1,\alpha}\,A_{2,\alpha})^j\,A_{1,\alpha}\,(1-k_\alpha)\,r_\alpha, \qquad j = 1, 2, \ldots, \text{ for every } \alpha \in \Lambda$$

and

$$d_\alpha(x_{2j}, x_{2j+1}) \le (A_{1,\alpha}\,A_{2,\alpha})^j\,(1-k_\alpha)\,r_\alpha, \qquad j = 2, 3, \ldots, \text{ for every } \alpha \in \Lambda.$$

Now it is immediate since $0 < A_{1,\alpha}\,A_{2,\alpha} < 1$ that (x_n) is a Cauchy sequence, and hence, converges to $x \in B(x_0, r)$ since X is complete. It remains to show $x \in F(x)$. Notice for $\alpha \in \Lambda$ that

$$\begin{aligned} \operatorname{dist}_\alpha(x, F(x)) &\le d_\alpha(x, x_{2n+1}) + \operatorname{dist}_\alpha(x_{2n+1}, F(x)) \\ &\le d_\alpha(x, x_{2n+1}) + D_\alpha(F(x_{2n}), F(x)), \end{aligned}$$

and so

$$\begin{aligned} D_\alpha\left(F(x_{2n}), F(x)\right) &\le a_{1,\alpha} \operatorname{dist}_\alpha\left(x_{2n}, F(x_{2n})\right) + a_{2,\alpha} \operatorname{dist}_\alpha\left(x, F(x)\right) + a_{3,\alpha} \operatorname{dist}_\alpha\left(x, F(x_{2n})\right) \\ &\quad + a_{4,\alpha} \operatorname{dist}_\alpha\left(x_{2n}, F(x)\right) + a_{5,\alpha}\, d_\alpha(x_{2n}, x) \\ &\le a_{1,\alpha}\, d_\alpha\left(x_{2n}, x_{2n+1}\right) + a_{2,\alpha}\left[d_\alpha\left(x, x_{2n+1}\right) + D_\alpha\left(F(x_{2n}), F(x)\right)\right] \\ &\quad + a_{3,\alpha}\, d_\alpha\left(x, x_{2n+1}\right) + a_{4,\alpha}\left[d_\alpha(x_{2n}, x_{2n+1}) + D_\alpha\left(F(x_{2n}), F(x)\right)\right] \\ &\quad + a_{5,\alpha}\, d_\alpha(x_{2n}, x). \end{aligned}$$

Consequently, for each $x \in \Lambda$, we have

$$\begin{aligned} D_\alpha\left(F(x_{2n}), F(x)\right) \le &\left(\frac{a_{1,\alpha} + a_{4,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x_{2n}, x_{2n+1}\right) \\ &+ \left(\frac{a_{2,\alpha} + a_{3,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x, x_{2n+1}\right) \\ &+ \left(\frac{a_{5,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x_{2n}, x\right). \end{aligned}$$

As a result, we have for each $x \in \Lambda$ that

$$\begin{aligned} \operatorname{dist}_\alpha\left(x, F(x)\right) \le &\left(\frac{a_{1,\alpha} + a_{4,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x_{2n}, x_{2n+1}\right) \\ &+ \left(\frac{1 + a_{3,\alpha} + a_{4,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x, x_{2n+1}\right) \\ &+ \left(\frac{a_{5,\alpha}}{1 - a_{2,\alpha} - a_{4,\alpha}}\right) d_\alpha\left(x_{2n}, x\right). \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $x \in \overline{F(x)} = F(x)$ and we are finished. ∎

A special case of Theorem 2.1 is the following fixed-point result.

COROLLARY 2.2. *Let X be a complete gauge space and $F : X \to C(X)$. Suppose there exist constants $a_1 = \{a_{1,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_2 = \{a_{2,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_3 = \{a_{3,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_4 = \{a_{4,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, and $a_5 = \{a_{5,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ such that for every $\alpha \in \Lambda$ and every x, $y \in X$, we have*

$$\begin{aligned} D_\alpha(F(x), F(y)) \le\; & a_{1,\alpha} \operatorname{dist}_\alpha\left(x, F(x)\right) + a_{2,\alpha} \operatorname{dist}_\alpha\left(y, F(y)\right) + a_{3,\alpha} \operatorname{dist}_\alpha\left(y, F(x)\right) \\ &+ a_{4,\alpha} \operatorname{dist}_\alpha\left(x, F(y)\right) + a_{5,\alpha}\, d_\alpha(x, y), \end{aligned}$$

where for every $\alpha \in \Lambda$, $a_{1,\alpha}+a_{2,\alpha}+a_{3,\alpha}+a_{4,\alpha}+a_{5,\alpha} < 1$, $a_{1,\alpha}+a_{4,\alpha}+a_{5,\alpha} > 0$, $a_{2,\alpha}+a_{3,\alpha}+a_{5,\alpha} > 0$, and with either $a_{1,\alpha} = a_{2,\alpha}$ or $a_{3,\alpha} = a_{4,\alpha}$. In addition, assume the following condition is satisfied:

$$\begin{aligned} &\text{for every } x \in X \text{ and every } \epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda \text{ there exists } y \in F(x) \\ &\text{with } d_\alpha\left(x, y\right) \le \operatorname{dist}_\alpha\left(x, F(x)\right) + \epsilon_\alpha \text{ for every } \alpha \in \Lambda. \end{aligned} \tag{2.9}$$

Then F has a fixed point.

PROOF. Fix $x_0 \in X$. For every $\alpha \in \Lambda$, choose $r_\alpha > 0$ so that

$$\operatorname{dist}_\alpha\left(x_0, F(x_0)\right) < \left(1 - \frac{A_{1,\alpha}\left(1 + A_{2,\alpha}\right)}{1 + A_{1,\alpha}}\right) r_\alpha.$$

Now Theorem 2.1 guarantees that there exists $x \in B(x_0, r)$ with $x \in F(x)$. ∎

Next we obtain a homotopy type result for multivalued contractions of Bose, Mukherjee type.

THEOREM 2.3. *Let X be a complete gauge space with U an open subset of X. Suppose $H : \overline{U} \times [0,1] \to C(X)$ is a closed map (i.e., has closed graph) with the following satisfied:*

(a) *$x \notin H(x,t)$ for $x \in \partial U$ and $t \in [0,1]$;*

(b) *there exist constants $a_1 = \{a_{1,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_2 = \{a_{2,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_3 = \{a_{3,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, $a_4 = \{a_{4,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$, and $a_5 = \{a_{5,\alpha}\}_{\alpha\in\Lambda} \in [0,1)^\Lambda$ with for all $t \in [0,1]$, and every $\alpha \in \Lambda$ and $x, y \in \overline{U}$ we have $D_\alpha(H(x,t), H(y,t)) \le a_{1,\alpha} \operatorname{dist}_\alpha(x, H(x,t)) + a_{2,\alpha} \operatorname{dist}_\alpha(y, H(y,t)) + a_{3,\alpha} \operatorname{dist}_\alpha(y, H(x,t)) + a_{4,\alpha} \operatorname{dist}_\alpha(x, H(y,t)) + a_{5,\alpha}\, d_\alpha(x,y)$ with for every $\alpha \in \Lambda$, $a_{1,\alpha}+a_{2,\alpha}+a_{3,\alpha}+a_{4,\alpha}+a_{5,\alpha} < 1$, $a_{1,\alpha}+a_{4,\alpha}+a_{5,\alpha} > 0$, $a_{2,\alpha}+a_{3,\alpha}+a_{5,\alpha} > 0$, and with either $a_{1,\alpha} = a_{2,\alpha}$ or $a_{3,\alpha} = a_{4,\alpha}$;*

(c) *for every $t \in [0,1]$, for every $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ there exists $y \in H(x,t)$ with $d_\alpha(x,y) \le \operatorname{dist}_\alpha(x, H(x,t)) + \epsilon_\alpha$ for every $\alpha \in \Lambda$; and*

(d) *there exists $M \in (0,\infty)^\Lambda$ and there exists a continuous increasing function $\phi : [0,1] \to \mathbf{R}$ such that $D_\alpha(H(x,t), H(x,s)) \le M_\alpha |\phi(t) - \phi(s)|$ for all t, $s \in [0,1]$ and $x \in \overline{U}$, for every $\alpha \in \Lambda$.*

Then $H(\,.\,,0)$ has a fixed point iff $H(\,.\,,1)$ has a fixed point.

PROOF. Suppose $H(\,.\,,0)$ has a fixed point. Consider

$$Q = \{(t,x) \in [0,1] \times U : x \in H(x,t)\}.$$

Now Q is nonempty since $H(\,.\,,0)$ has a fixed point. On Q define the partial order

$$(t,x) \le (s,y) \text{ iff } t \le s \qquad \text{and} \qquad d_\alpha(x,t) \le 2\left(\frac{1+A_{1,\alpha}}{1-A_{1,\alpha}\,A_{2,\alpha}}\right) M_\alpha\,[\phi(s)-\phi(t)] \qquad \text{for every } \alpha \in \Lambda,$$

where $A_{1,\alpha}$ and $A_{2,\alpha}$ are as in Theorem 2.1. Let P be a totally ordered subset of Q and let

$$t^\star = \sup\{t : (t,x) \in P\}.$$

Take a sequence $\{(t_n,x_n)\}$ in P such that $(t_n,x_n) \le (t_{n+1},x_{n+1})$ and $t_n \to t^\star$. We have

$$d_\alpha(x_m,x_n) \le 2\left(\frac{1+A_{1,\alpha}}{1-A_{1,\alpha}\,A_{2,\alpha}}\right) M_\alpha\,[\phi(t_m)-\phi(t_n)], \qquad \text{for all } m > n \text{ and every } \alpha \in \Lambda,$$

and so (x_m) is a Cauchy sequence, which converges to some $x^\star \in \overline{U}$. Now since H is a closed map we have $(t^\star, x^\star) \in Q$ (note $x^\star \in H(x^\star, t^\star)$ by closedness and (a) implies $x^\star \in U$). It is also immediate from the definition of $t^\star$ and the fact that P is totally ordered that

$$(t,x) \le (t^\star, x^\star), \qquad \text{for every } (t,x) \in P.$$

Thus, $(t^\star,x^\star)$ is an upper bound of P. By Zorn's Lemma Q admits a maximal element $(t_0,x_0) \in Q$.

We *claim* $t_0 = 1$ (if our claim is true then we are finished). Suppose our claim is false. Then, choose $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0,\infty)^\Lambda$ and $t \in (t_0,1]$ with

$$B(x_0,r) \subseteq U \qquad \text{and} \qquad r_\alpha = 2M_\alpha\left(\frac{1+A_{1,\alpha}}{1-A_{1,\alpha}\,A_{2,\alpha}}\right)[\phi(t)-\phi(t_0)], \qquad \text{for every } \alpha \in \Lambda.$$

Notice for every $\alpha \in \Lambda$ that

$$\begin{aligned}\operatorname{dist}_\alpha(x_0, H(x_0,t)) &\le \operatorname{dist}_\alpha(x_0, H(x_0,t_0)) + D_\alpha(H(x_0,t_0)), H(x_0,t))) \\ &\le M_\alpha\,[\phi(t)-\phi(t_0)] = \frac{1}{2}\left(\frac{1-A_{1,\alpha}\,A_{2,\alpha}}{1+A_{1,\alpha}}\right) r_\alpha < \left(\frac{1-A_{1,\alpha}\,A_{2,\alpha}}{1+A_{1,\alpha}}\right) r_\alpha.\end{aligned}$$

Now Theorem 2.1 guarantees that $H(\,.\,,t)$ has a fixed point $x \in B(x_0,r)$. Thus, $(x,t) \in Q$ and notice since

$$d_\alpha(x_0,x) \le r_\alpha = 2M_\alpha\left(\frac{1+A_{1,\alpha}}{1-A_{1,\alpha}\,A_{2,\alpha}}\right)[\phi(t)-\phi(t_0)], \qquad \text{for every } \alpha \in \Lambda, \quad \text{and} \quad t_0 < t,$$

we have $(t_0,x_0) < (t,x)$. This contradicts the maximality of (t_0,x_0). ∎

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